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EVOLUTIONARY RANDOM RESPONSES OF LINEAR RANDOM STRUCTURES

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1. INTRODUCTION

Indeterminate factors often appear in engineering structures. In fact, a mathematical model is just an abstract representation of a real structure, which can never be exactly the same as the real structure. For instance, measurements cannot be absolutely accurate, manufacturing processes cannot be perfect, mechanical properties of materials cannot be exactly identified, and so forth. These indeterminate factors are often modelled as random variables. Structural models with random variables as their parameters are often called random structures. Hence, there is a need for statistical analyses of responses of random structures. On the other hand, indeterminacy also exists in excitations, especially in earthquake excitations. Therefore, earthquake excitations are often modelled as random processes, mostly evolutionary ones.

The mathematical tool for solving response problems of random structures is quite difficult and far from maturity, especially when random dynamic loads are involved. Therefore, most published works on random vibrations of random structures are restricted to the statistical properties of eigenvalue problems or responses to static loads [1].

However, one of the important features of a random structure is that its sample system is a deterministic one, whether time-invariant or time-variant. The unified approach to evolutionary random response problems, suggested by Fang *et al.* [2, 3], is applicable to these sample systems, too. While studying the evolutionary random response problems of random structures through Monte-Carlo simulations, the computational effort can be greatly reduced by applying the unified approach to the sample systems. In what follows, the ways to deal with evolutionary random response problems of a random structure under earthquake excitations are explained.

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2. A UNIFIED APPROACH TO EVOLUTIONARY RANDOM RESPONSE PROBLEMS

The differential equation of the response of a discrete random structure, in general, may be expressed as follows:

$$M\ddot{x} + C\dot{x} + Kx = bf(t), \tag{1}$$

where the response $x(t)$ is an n-vector with real, time-function elements; M, C, K are $n \times n$ matrices with random variable elements, among which the sample matrices of M are supposed to be positive-definite and statistical properties of all the random elements are given; b is an n-vector with real constant elements; and $f(t)$ is a scalar excitation, either a deterministic or a random one.

Since a sample system of a random structure is a deterministic one, for the time being supposed to be time-invariant, the response of a sample system under zero initial conditions may be expressed as

$$x(t) = \int_0^t h(u)bf(t-u)\mathrm{d}u. \tag{2}$$

When the excitation $f(t)$ is deterministic, so is the response $x(t)$. The cross product of $x(t)$ is simply

$$x(t)x^{\mathrm{T}}(t) = \int_0^t \int_0^t h(u)bf(t-u)f(t-v)b^{\mathrm{T}}h^{\mathrm{T}}(v)\,\mathrm{d}u\mathrm{d}v. \tag{3}$$

Now suppose that the excitation $f(t)$ is an evolutionary random process with a zero mean and an evolutionary covariance function, which can be expressed as

$$C_f(t_1, t_2) = E[f(t_1)f(t_2)] = \int_{-\infty}^{\infty} A(\omega, t_1)\,\bar{A}(\omega, t_2)\mathrm{e}^{\mathrm{j}\omega(t_2-t_1)}S(\omega)\mathrm{d}\omega, \tag{4}$$

where $S(\omega)$ is the power spectral density of a stationary random process, and $A(\omega, t)$ is a deterministic modulating function, which regulates the stationary random process into the evolutionary random excitation $f(t)$. Since the response of a sample system to a sample excitation under zero initial conditions still takes the form of equation (2), the mean response $E[x\,(t)]$ is a zero vector. Meanwhile, by equation (4), the covariance matrix of the evolutionary random response of a sample system should be

$$C_x(t,t) = E[x(t)x^{\mathrm{T}}(t)] = \int_0^t \int_0^t h(u)bE[f(t-u)f(t-v)]b^{\mathrm{T}}h^{\mathrm{T}}(v)\mathrm{d}u\mathrm{d}v$$

$$\equiv \int_{-\infty}^{\infty} G(\omega, t)\bar{G}^{\mathrm{T}}(\omega, t)S(\omega)\mathrm{d}\omega \tag{5}$$

where

$$G(\omega, t) = \int_0^t h(u)bA(\omega, t-u)\mathrm{e}^{-\mathrm{j}\omega(t-u)}\mathrm{d}u. \tag{6}$$

It is easy to see from equation (2) that $G(\omega, t)$ is just the deterministic transient response of the sample system under zero initial conditions to a deterministic excitation $bA(\omega, t)\mathrm{e}^{-\mathrm{j}\omega t}$ with ω as a fixed parameter. For any deterministic linear system, whether time-invariant or time-variant, it is easy to obtain those deterministic transient responses for different ω's by any existing effective numerical method, such as the Runge–Kutta method. Once all the $G(\omega, t)$'s are obtained, the covariance matrix $C_x(t, t)$ is at hand.

As long as the statistical properties of the random structure are independent of the statistical properties of the random excitations, the covariance matrix of the evolutionary random response of the random structure can be obtained by

$$E_s[C_x(t, t)] = E_s\left[\int_{-\infty}^{\infty} G(\omega, t)\bar{G}^{\mathrm{T}}(\omega, t)S(\omega)\mathrm{d}\omega \right] \tag{7}$$

where E_s means taking the ensemble average with respect to the random structure only. A straightforward method suitable for this job is the Monte-Carlo method.

In case the random parameters of a random structure can be expressed as the series form of a certain small parameter, there is an alternative method, via the perturbation technique to deal with the response problems of a random structure. By the perturbation technique [4], the response problem of a random structure subject to external excitations can be reduced to a series of problems of a deterministic structure subject to successively derived random excitations. Then, the above-stated unified approach to evolutionary random response problems can give full play to its effectiveness hereafter.

3. EVOLUTIONARY RANDOM RESPONSE OF SHEAR BEAM UNDER EARTHQUAKE EXCITATION

Consider a uniform shear beam with length L, shear stiffness k, damping coefficient η, and mass per unit length ρ as shown in Figure 1. The beam is fixed at the base with co-ordinate $x = 0$ and is free at the end $x = L$. When the base is given an arbitrary transient acceleration $\ddot{y}_0(t)$, the relative transverse displacement $y(x, t)$ at section x is governed by the following partial differential equation:

$$\frac{\partial^2 y}{\partial t^2} - c\frac{\partial^3 y}{\partial t \partial x^2} - \alpha^2 \frac{\partial^2 y}{\partial x^2} = -\ddot{y}_0(t), \tag{8}$$

where the wave speed $\alpha = (k/\rho)^{1/2}$ and the reduced damping coefficient $c = \eta/\rho$. The boundary conditions are $y = 0$, when $x = 0$; and $\partial y/\partial x = 0$ when $x = L$. In the following paragraphs, the section co-ordinate x is replaced by the relative co-ordinate $z = x/L$,

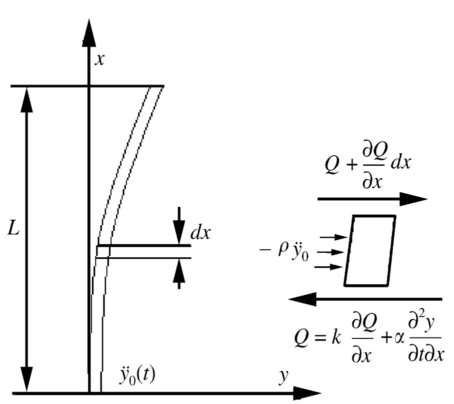

Figure 1. A uniform shear beam model.

which may be convenient for comparison. Then, equation (8) reduces to

$$\frac{\partial^2 y}{\partial t^2} - \frac{c}{L^2}\frac{\partial^3 y}{\partial t \partial z^2} - \frac{\alpha^2}{L^2}\frac{\partial^2 y}{\partial z^2} = -\ddot{y}_0(t) \tag{9}$$

and the boundary conditions reduce to $y = 0$, when $z = 0$; and $\partial y/\partial z = 0$, when $z = 1$.

While looking for a modal solution as a sum of the products of the normal modes $\varphi_i(z)$ and normal co-ordinates $Y_i(t)$, we can put

$$y(z, t) = \sum_{i=1}^{\infty} \varphi_i(z) Y_i(t). \tag{10}$$

The normal modes for the shear beam, $\varphi_i(z) = \sin(L/\alpha)\,\omega_i z$, $i = 1,2,\ldots$, with the natural frequencies $\omega_i = (2i-1)\,\pi\alpha/2L$, satisfy the free vibration relations and the required orthogonality conditions. Then, by using the orthogonal conditions for normal modes, equation (9) can be finally reduced into a set of the second order ordinary differential equations for the independent modal systems:

$$\ddot{Y}_i + 2\xi_i\omega_i\dot{Y}_i + \omega_i^2 Y_i = G_i(t) \quad i = 1,2,\ldots, \tag{11}$$

where

$$\xi_i = \frac{1}{2}\frac{c}{\alpha^2}\omega_i = \frac{(2i-1)\pi}{4}\frac{c}{\alpha L}$$

and

$$G_i(t) = 2\int_0^1 -\ddot{y}_0(t)\,\varphi_i(z)\mathrm{d}z = \beta_i\ddot{y}_0(t), \quad \beta_i = -\frac{4}{(2i-1)\pi}. \tag{12}$$

The particular solutions of equation (11) for zero initial conditions can be obtained by Duhamel integrals

$$Y_i(t) = \int_0^t G_i(t-u)\,h_i(u)\mathrm{d}u \tag{13}$$

where $h_i(u)$ is the impulse response of the ith modal system. Correspondingly, the relative transverse displacement $y(z, t)$ for a sample beam is

$$y(z, t) = \sum_{i=1}^{\infty} \varphi_i(z) Y_i(t) = \sum_{i=1}^{\infty} \beta_i\varphi_i(z)\int_0^t \ddot{y}_0(t-u)h_i(u)\mathrm{d}u. \tag{14}$$

The ground acceleration, $\ddot{y}_0(t)$, takes the evolutionary random excitation model for the Niigata earthquake. Suppose that the mean value of the random excitation is zero, and the evolutionary power spectrum is expressed as [5]

$$S_{\ddot{y}_0}(\omega, t) = |A(\omega, t)|^2 S_0$$

with

$$A(\omega, t) = \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{\max(\mathrm{e}^{-at} - \mathrm{e}^{-bt})}\left\{\frac{\Omega^4(t) + 4\zeta^2(t)\Omega^2(t)\omega^2}{[\omega^2 - \Omega^2(t)]^2 + 4\zeta^2(t)\Omega^2(t)\omega^2}\right\}^{1/2}$$

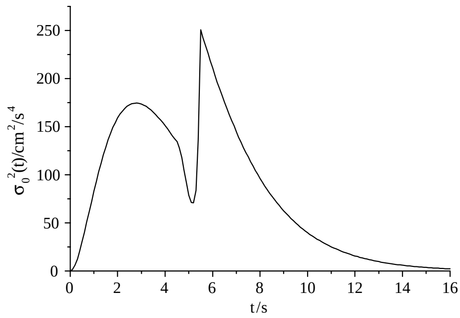

Figure 2. Time-dependent variance of ground acceleration, $\sigma_0^2(t)$.

and

$$\zeta(t) = \begin{cases} 0{\cdot}64 & 0 \leqslant t \leqslant 4{\cdot}5 \text{ s}, \\ 1{\cdot}25(t-4{\cdot}5)^3 - 1{\cdot}875(t-4{\cdot}5)^2 + 0{\cdot}64 & 4{\cdot}5 \leqslant t - \leqslant 5{\cdot}5 \text{ s}, \\ 0{\cdot}015 & t \leqslant 5{\cdot}5 \text{ s}, \end{cases}$$

$$\Omega(t) = \begin{cases} 15{\cdot}56 \text{ rad/s}, & 0 \leqslant t \leqslant 4{\cdot}5 \text{ s}, \\ 27{\cdot}12(t-4{\cdot}5)^3 - 40{\cdot}68(t-4{\cdot}5)^2 + 15{\cdot}56 \text{ rad/s}, & 4{\cdot}5 \leqslant t \leqslant 5{\cdot}5 \text{ s}, \\ 2 \text{ rad/s}, & t \geqslant 5{\cdot}5 \text{ s}. \end{cases}$$

Taking $S_0 = 2\ \text{cm}^2/\text{s}^3$, $a = 0{\cdot}25/\text{s}^{-1}$, and $b = 0{\cdot}5/\text{s}^{-1}$, the time-dependent variance of ground acceleration, $\sigma_0^2(t)$, is plotted in Figure 2.

Then, the mean square evolutionary random response of a sample beam at section z can be obtained as

$$E[y^2(z,t)] = \int_{-\infty}^{\infty} S_0 \left\{ \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_i \beta_k \varphi_i(z) \varphi_k(z) G_i(\omega, t) \bar{G}_k(\omega, t) \right\} \mathrm{d}\omega, \tag{15}$$

where

$$G_i(\omega, t) = \int_0^t A(\omega, t-u) \mathrm{e}^{\mathrm{j}\omega(t-u)} h_i(u) \mathrm{d}u.$$

Once all the $G_i(\omega, t)$'s are found, the mean square response $E[y^2(z,t)]$ is at hand. Since the statistical properties of the random structure are supposed to be independent of those of the random excitation, the mean square evolutionary random response of the random structure can be obtained by further taking the ensemble average with respect to the random structure.

In the following calculation, the parameters ρ, k and η are supposed to be deterministic and are given as

$$\rho = 1 \times 10^5 \text{ kg/m}, \quad k = 2 \times 10^9 \text{ kg m/s}^2, \quad \eta = 3{\cdot}125 \times 10^8 \text{ kg m/s}$$

and the only indeterminate parameter of the structure is the length L, supposed to be a random variable with truncated normal distribution, $L \in [97, 103]$ m, and with its mean

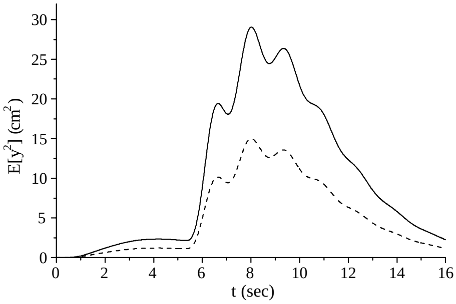

Figure 3. Mean square responses of sample beams. ---- A; —— B.

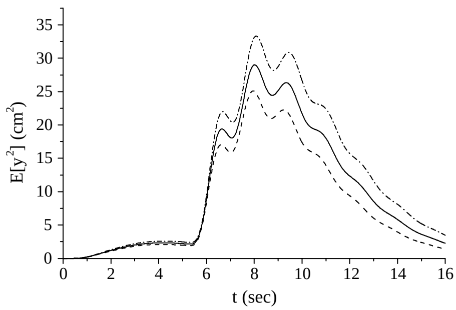

Figure 4. Mean square responses of the random beam. ---- A; —— B; -·-·-· C.

value and standard deviation, respectively, as

$$E[L] = 100 \text{ m}, \quad \sigma[L] = 1 \text{ m}.$$

Note that in equation (15), the mean square evolutionary random response is expressed in the form of a sum of infinite terms. However, the only possible way in practical computation is to truncate the sum by finite terms. By the definition in equation (12), the β_i's in equation (15) decrease rapidly with the increase of i. It is found in calculation that the final results are accurate enough by preserving only the first five terms in the summation of modal responses.

Some of the numerical results by Monte–Carlo simulations are shown in Figures 3 and 4. In Figure 3, the mean square evolutionary random responses at different sections for a sample beam with length $L = 100$ m are given, where curve A for beam section $z = 0{\cdot}5$, and curve B for beam section $z = 1$. In Figure 4, the ensemble mean square evolutionary random response of the random beam at section $z = 1$ is given by the solid curve B. For comparison, the mean square evolutionary random responses at section $z = 1$ for two extreme sample beams with $L = 97$ m and 103 m are also given in Figure 4 by curves A and C respectively. It can be seen that although the maximum relative deviation of beam length is only 3%, the maximum relative deviation of the mean square response is about 12%.

4. CONCLUSIONS

(1) The unified approach to evolutionary random response problems can be applied to random structures as well. Numerical examples show that the computational effort for

Monte–Carlo simulation analysis of random structural responses can be greatly reduced by combining with the unified approach.

(2) Although illustrated by a relatively simple random beam example, the method can also be applied to more complicated random structures with multiple random paramters, or even with their sample structures as deterministic time-variant systems.

(3) The unified approach can be applied to random response problems of random structures in an alternative way, namely to combine the approach with perturbation techniques. The related results will be reported later.

(4) From the derivation, one can see that the unified approach to evolutionary random response problems is completely based on the classical random vibration theory. Hence, it is better to say that the unified approach is simply a logical outcome of better understanding of the classical random vibration theory rather than a new method. However, it is really a very efficient and useful approach to evolutionary random response problems, without resorting to some advanced and complicated mathematical tools.

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